B.Sc. Part-I
Physics

Langrange's Equation of motion

In this we shall obtain equations which will give us the whole motion of the system.

The will be obtained in terms of any co-ordinates that we may find convenient to use, the word co-ordinates being extended to mean any independent quantities which, when they are given, determined the position of the body or bodies under consideration,

Let (x, y, z) be the co-ordinates of any particle on of the system reffered rectangular axes, and let them be expressed in terms of a certain number of independent variables θ, ϕ, ψ .... so that, if t be the time, we have

$$x = f(t, \theta, \phi, \psi \cdots\cdots) \quad \text{——— (1)}$$

with simillar expressions for y and z.

These equations are not to contain θ, ϕ .... or any other differential coefficient with regard to the time.

As usual, let dots denote differential co-efficients with regard to the time, and let

$\frac{dx}{d\theta}, \frac{dx}{d\phi}$ .... denote partial differental coeff.

The differentiating (1) we have

$$\dot{x} = \frac{dx}{dt} + \frac{dx}{d\theta}\cdot\dot{\theta} + \frac{dx}{d\phi}\cdot\dot{\phi} + \cdots\cdots \quad (2)$$

On differentiating (2) partially with regard to θ we have

$$\frac{d\dot{x}}{d\theta} = \frac{dx}{d\theta} \quad \text{.......... (3)}$$

Again differentiating (2) with regard to θ, we have

$$\frac{d\dot{x}}{d\theta} = \frac{d^2x}{d\theta dt} + \frac{d^2x}{d\theta^2}\cdot\theta + \frac{d^2x}{d\theta d\phi}\cdot\phi + \cdots\cdots$$

$$= \frac{d}{dt}\left[\frac{dx}{d\theta}\right] \quad \text{———— (4)}$$

If T be the kinetic energy of the system, then

$$T = \frac{1}{2}\Sigma m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad \text{...... (5)}$$

Now the reversed effective forces and the impressed forces form a system of forces in equillibrium, so that their equation of virtual work vanishes.

The first of those for a vibration of θ only,

$$= \Sigma m\left[\ddot{x}\frac{dx}{d\theta} + \ddot{y}\frac{dy}{d\theta} + \ddot{z}\frac{dz}{d\theta}\right]\delta\theta$$

$$= \frac{d}{dt}\Sigma m\left[\dot{x}\frac{dx}{d\theta} + \cdots\cdots + \cdots\cdots\right]\delta\theta$$

$$- \Sigma m\left[\dot{x}\frac{d}{dt}\left(\frac{dx}{d\theta} + \cdots\cdots + \cdots\cdots\right]\delta\theta$$

$$= \frac{d}{dt}\Sigma\left[\dot{x}\frac{d\dot{x}}{d\theta} + \cdots\cdots + \cdots\cdots\right]\delta\theta$$

$$- \Sigma m\left[\dot{x}\frac{d\dot{x}}{d\theta} + \cdots\cdots + \cdots\cdots\right]\delta\theta$$

by equation (3) and (4)

$$= \frac{d}{dt} \cdot \frac{d}{d\dot{\theta}} \sum m \cdot \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \delta\theta - \frac{d}{d\theta} \sum m \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \delta\theta$$

$$= \left[\frac{d}{dt} \frac{dT}{d\dot{\theta}} - \frac{dT}{d\theta} \right] \delta\theta \quad \text{by equation (5)} \quad \text{———} \quad (6)$$

Again if V be the work-function, we have the virtual work of the impressed forces, for a variation of θ alone,

$$= \sum m \left[X \frac{dx}{d\theta} + Y \frac{dy}{d\theta} + Z \frac{dz}{d\theta} \right] \delta\theta$$

$$= \left[\frac{dV}{dx} \frac{dx}{d\theta} + \frac{dV}{dy} \frac{dy}{d\theta} + \frac{dV}{dz} \frac{dz}{d\theta} \right] \delta\theta$$

$$= \frac{dV}{d\theta} \cdot \delta\theta \quad \text{———} \quad (7)$$

Equating (6) and (7), we have

$$\frac{d}{dt} \left(\frac{dT}{d\dot{\theta}} \right) - \frac{dT}{d\theta} = \frac{dV}{d\theta} \quad \text{———} \quad (8)$$

Similarly, we have the equations

$$\frac{d}{dt} \left(\frac{dT}{d\dot{\phi}} \right) - \frac{dT}{d\phi} = \frac{dV}{d\phi}$$

$$\frac{d}{dt} \left(\frac{dT}{d\dot{\phi}} \right) - \frac{dT}{d\psi} = \frac{dV}{d\psi}$$

and so on, there being one equation corresponding to each independent co-ordinate system.

These equations are known as Langranges' equations in Generalised Coordinates.

COR. If K be the potential energy of the system, Since V = a constant − K, equation (8) becomes

$$\frac{d}{dt}\left(\frac{dT}{d\theta}\right) - \frac{dT}{d\theta} + \frac{dK}{d\theta} = 0$$

If we put T − K = L, so that L is equal to the difference between the kinetic and potential energies the since V does not contain θ, ϕ, etc this equation can be written in the form

$$\frac{d}{dt}\left[\frac{dL}{d\theta}\right] - \frac{dL}{d\theta} = 0$$

L is called the Langrangian Function.

When a system is such that the co-ordinates of any particle of it can be expressed in terms of independent co-ordinates by equations which we do not contain differential ~~equation~~ coefficient with regard to the time, the system is said to be holonomous.

Continue